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Minimal Change — A Criterion for Choosing between Competing Models —

by
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Minimal Change
- A Criterion for Choosing between Competing Models -

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Abstract

We often encounter a situation in which we are forced to make some decision even if there is not enough information. In that situation, we normally use common sense to draw a conclusion from the incomplete knowledge. This reasoning mechanism can be regarded as meta-reasoning which chooses the preferred models from several consistent models. This paper formalizes common sense reasoning of tree-structured inheritance systems and temporal projection into one framework, *minimal change*. Both types of reasoning have a common mechanism to prefer a model which changes minimally in one direction. In inheritance systems, the direction is from superclass to subclass, and in temporal projection, the direction is from earlier state to later state.

1. Introduction

To formalize common sense reasoning is a major problem in artificial intelligence because it expresses the human way of making a decision when there is not enough information. The results from that reasoning are not logically true, but it often works surprisingly well because common sense is a collection of normal results.

We cannot formalize it in a classical logical way since classical logic is monotonic in the sense that the derived results increase as more information is added, while common sense reasoning is *nonmonotonic*. It sometimes produces false conclusions, and when a result is found to be false, then that result is removed. Therefore, we must formalize it in a different way.

Along this idea, several formalisms of non-monotonic reasoning are provided [McDermott80, McCarthy80, and Reiter80]. [McDermott80] and [Reiter80] are types of logic which incorporates the notion of unprovability. [McCarthy80] formalizes that people tend to think that given information is only true.

Unfortunately, their research is not applicable to reasoning of inheritance systems and temporal projection. [McCarthy84] points out that a simple abnormality formalism does not work by simple *circumscription* in inheritance systems and introduces *prioritized circumscription*. [Etherington87] also points out that the *normal default* *theory* does not produce a unique *extension* in the inheritance system and introduces the *semi-normal default theory*. [Hanks86] points out that simple *circumscription*, *normal default theory*, and *NML-I* are not applicable to the temporal projection problem (details are given in [Hanks87]), and several people suggest solutions [Kautz86, Lifschitz86, Shoham86 and more in Frame87].

The formalism presented here is another solution to these problems. Our approach is providing a preference criterion of consistent models in these problems. These problems are identified with reasoning in tree-structured multiple worlds, and common sense reasoning of these problems is regarded as selecting a preferred model from consistent models. A criterion of selection of a model called *minimal change* is formalized. *Minimal change* means that our common sense prefers a model which has less change from a current world to a new world.

2. Nonmonotonic Reasoning

As several people point out, circumscription, normal default theory and NML-1 are not applicable to such types of common sense reasoning as inheritance system and temporal projection. We see the details by using normal default theory. In the default theory, there are some *defaults* additional to axioms. Default is any expression of the form:

$$\frac{\alpha : \mathbf{M}\beta_1, ..., \mathbf{M}\beta_m}{w}$$

where $\alpha, \beta_1, ..., \beta_m, w$ are wffs. The meaning of **M** is informally understood as "it is consistent to assume".

Normal default is a default of the following form.

$$\frac{\alpha : \mathbf{M}w}{w}$$

Normal default theory consists of axioms and normal defaults.

An *extension* is a set of derived result from default theory and should be satisfied with the following properties.

(1) It should contain set of axioms.
(2) It should be closed under logical consequence.
(3) Let $(\alpha : \mathbf{M}\beta_1, ..., \mathbf{M}\beta_m/w)$ be a default. If an extension includes α and does not include $\neg\beta_1, ..., \neg\beta_m$, then an extension must include w.

2.1 Tree-structured Inheritance System

We use the following example.

(1) Animals do not normally fly.
(2) Birds are animals but normally fly.
(3) Penguins are birds but do not normally fly.
(4) Fish are animals and normally swim.

(5) Mammals are animals.
(6) Bats are mammals but normally fly.

The above information seems to be expressed as the following axioms and defaults.

$$\text{BIRD} \supset \text{ANIMAL} \tag{2.1.1}$$
$$\text{PENGUIN} \supset \text{BIRD} \tag{2.1.2}$$
$$\text{FISH} \supset \text{ANIMAL} \tag{2.1.3}$$
$$\text{MAMMAL} \supset \text{ANIMAL} \tag{2.1.4}$$
$$\text{BAT} \supset \text{MAMMAL} \tag{2.1.5}$$
$$\frac{\text{ANIMAL} : \text{M}\neg\text{FLY}}{\neg\text{FLY}} \tag{2.1.6}$$
$$\frac{\text{BIRD} : \text{MFLY}}{\text{FLY}} \tag{2.1.7}$$
$$\frac{\text{PENGUIN} : \text{M}\neg\text{FLY}}{\neg\text{FLY}} \tag{2.1.8}$$
$$\frac{\text{FISH} : \text{MSWIM}}{\text{SWIM}} \tag{2.1.9}$$
$$\frac{\text{BAT} : \text{MFLY}}{\text{FLY}} \tag{2.1.10}$$

However the above axioms and defaults do not work, because for example suppose that BIRD is asserted, there are two extensions which include the following facts.

$$E_1 \supset \{\text{BIRD}, \text{FLY}, \text{ANIMAL}\}$$

$$E_2 \supset \{\text{BIRD}, \text{ANIMAL}, \neg\text{FLY}\}$$

The above set of facts in E_1 is informally obtained as follows. Since BIRD is in the extension, and it is consistent to assume FLY, FLY is in the extension by the default (2.1.7). Since BIRD is in the extension, ANIMAL is in the extension by the axiom (2.1.1).

The above set of facts in E_2 is informally obtained as follows. Since BIRD is in the extension, ANIMAL is in the extension by the axiom (2.1.1). Since ANIMAL is in the extension, and it is consistent to assume ¬FLY, ¬FLY is in the extension by the default (2.1.6). Note that in this case ¬FLY is in the extension, therefore it is not consistent to assume FLY, and FLY should not be in the extension.

Since E_1 includes FLY while E_2 includes ¬FLY, and there is no preference between them, we cannot tell whether birds fly or not while our intention says that birds fly.

2.2 Temporal Projection

We use the following example called the *Yale Shooting Problem* [Hanks86].

In the initial situation, S_0, the person is alive. The gun becomes loaded after an action, LOAD, is performed. If the gun is loaded then the person will not be alive after an action, SHOOT, is performed. Information about an action, WAIT, is not given. Then what facts do people think hold in the sequence of actions LOAD, WAIT and SHOOT ?

We use the *situation calculus*[McCarthy69] to express the above information. $T(F,S)$ expresses that the fact F is true in the situation S. The function $RESULT$ is the mapping from an action and a situation into another situation. For example, if S_0 is a situation and LOAD is an action, then $RESULT(\text{LOAD},S_0)$ is also a situation. If the fact, LOADED, is true in the situation after the action, LOAD, is performed from the situation, S_0, then we can express it as follows.

$T(\text{LOADED},RESULT(\text{LOAD},S_0))$

Then, the above information of the shooting scenario seems to be expressed as the following axioms and defaults.

$$T(\text{ALIVE},S_0) \tag{2.2.1}$$

$$\forall s[T(\text{LOADED},RESULT(\text{LOAD},s))] \tag{2.2.2}$$

$$\forall s[\neg T(\text{LOADED},s) \lor \neg T(\text{ALIVE},RESULT(\text{SHOOT},s))] \tag{2.2.3}$$

$$\frac{T(f,s) : \text{M}T(f,RESULT(a,s))}{T(f,RESULT(a,s))} \tag{2.2.4}$$

Informally, default (2.2.4) expresses that every fact, f, will hold continuously after every action, a, from every situation, s, if it is consistent to assume it.

However the above default theory does not work, because we can derive two extensions which include the following facts.

$E_1 \supset \{\ T(\text{ALIVE},S_0),\ T(\text{ALIVE},S_1),\ T(\text{LOADED},S_1),\ T(\text{ALIVE},S_2),$
$T(\text{LOADED},S_2),\ \neg T(\text{ALIVE},S_3)\ \}$

$E_2 \supset \{\ T(\text{ALIVE},S_0),\ T(\text{ALIVE},S_1),\ T(\text{ALIVE},S_2),\ T(\text{ALIVE},S_3),$
$T(\text{LOADED},S_1),\ \neg T(\text{LOADED},S_2)\ \}$

where $S_1 = RESULT(\text{LOAD},S_0)$ and $S_2 = RESULT(\text{WAIT},S_1)$ and $S_3 = RESULT(\text{SHOOT},S_2)$

The above set of facts in E_1 is informally obtained as follows. Since $T(\text{ALIVE},S_0)$ is in the extension, and it is consistent to assume $T(\text{ALIVE},S_1)$, it is in the extension by an instance of the default (2.2.4) where ALIVE, S_0 and LOAD are substituted for f, s and a respectively. Similarly $T(\text{ALIVE},S_2)$ is also in the extension. Since $T(\text{LOADED},S_1)$ is in the extension by (2.2.2), and it is consistent to assume $T(\text{LOADED},S_2)$, $T(\text{LOADED},S_2)$ is in the extension by the default. Then $\neg T(\text{ALIVE},S_3)$ is in the extension by (2.2.3).

The above set of facts in E_2 is informally obtained as follows.
$T(\text{ALIVE},S_0)$, $T(\text{ALIVE},S_1)$ and $T(\text{ALIVE},S_2)$ are in the extension in a similar way of

the above discussion. Then $T(\text{ALIVE},S_3)$ is in the extension by the default, because it is consistent to assume it. Then $\neg T(\text{LOADED},S_2)$ is in the extension by (2.2.3).

Since E_1 includes $\neg T(\text{ALIVE},S_3)$ while E_2 includes $T(\text{ALIVE},S_3)$, and there is no preference between them, we cannot tell whether the person will be alive or not after the actions LOAD, WAIT and SHOOT while the person will not be alive in our intended models.

3. Reasoning in Tree-Structured Multiple Worlds

We think that the above multiple extension problem arises from a lack of preference criterion which can not be expressed in those formalism such as circumscription, normal default theory and NML-1. Our solution is to map the above types of reasoning into tree-structured multiple worlds and to provide preference criterion between consistent models. This section shows how the above reasoning are mapped into tree-structured multiple worlds.

A *tree* is a directed graph with no cycles which satisfies the following conditions:

(1) There is only one node called the *root*, which has no entering edges.
(2) Every node except the root has exactly one entering edge.
(3) There is a unique path from the root to each node.

In the tree-structured inheritance system, classes correspond to the nodes and inheritance relations correspond to the edges. The root corresponds to the highest class. The direction is from superclass to subclass. The properties of classes are associated with the corresponding nodes. In the tree-structured inheritance system, once a property is given in a class, it is inherited in lower classes unless it is explicitly declared not to hold. It can be formalized that in this inheritance mechanism, the set of properties does not change with the inheritance relation until contradiction occurs, and if it occurs, the new set of properties changes minimally from the previous one to maintain consistency.

In temporal projection, states correspond to the nodes and actions correspond to the edges. The root corresponds to the initial state. The direction is from earlier state to later state. The facts of a state are associated with the corresponding nodes. In temporal projection, once a fact holds in a state, it continues to hold unless that fact is explicitly declared not to hold. It can be formalized that in this projection mechanism, the set of facts does not change with the time until contradiction occurs, and if it occurs, the set of facts are changed minimally from the previous one.

4. Formalism

As stated above, properties or facts are not absolutely true, but relative to a node. Therefore we use a predicate, $T(P,W)$ (in a similar way in the previous example of temporal projection), to express that a property or a fact, P, is true at a node, W. For simplicity, we use many-sorted logic. Variables $p_1,p_2,\ldots$ range over properties or facts and variables $w_1,w_2,\ldots$ range over nodes. We denote the parent node of node w other than the root by $last(w)$. Note that the parent node is unique, so we can define it as a function. We define partial order relation '$<$' over nodes: $w_1 < w_2$ iff there is a path from w_1 to w_2.

A *structure* M for a second-order language consists of a domain D, which is a non-empty set, and an interpretation function such that every n-ary function constant, F_n, is mapped into a function from D^n to D (written as $M[F_n]$) and every n-ary predicate constant, P_n, is mapped into a subset of D^n (written as $M[P_n]$). And, n-ary function variables range over any function from D^n to D and n-ary predicate variables range over any subset of D^n. $< t_1, ..., t_n >_M$ denotes an interpreted tuple where $t_1, ..., t_n$ are terms. If $P_n(t_1, ..., t_n)$ is true in M, we express this fact as $< t_1, ..., t_n >_M \in M[P_n]$. A *model* of a set of axioms is any structure M such that every formula in the set is true in M.

Now we define a relation $\preceq$ over models.

$M_1 \preceq M_2$ iff
(1) M_1 and M_2 have the same domain.
(2) Every function constant, and predicate constant except T receives the same interpretation in M_1 and M_2.
(3) The following statement is true(We omit M_1 of $< p, w >_{M_1}$ and M_2 of $< p, w >_{M_2}$, because $< p, w >_{M_1} = < p, w >_{M_2}$ by (2)).

$$
\begin{aligned}
&\forall p[< p, 0 > \in M_1[T] \equiv < p, 0 > \in M_2[T]] \wedge \\
&\forall w[\\
&\quad [0 < w \wedge \forall w'[0 < w' < w \supset \forall p[< p, w' > \in M_1[T] \equiv < p, w' > \in M_2[T]]]] \supset \\
&\quad \forall p[\\
&\qquad [< p, last(w) > \in M_2[T] \equiv < p, w > \in M_2[T]] \supset \\
&\qquad [< p, last(w) > \in M_1[T] \equiv < p, w > \in M_1[T]] \\
&\quad] \\
&]
\end{aligned}
$$

where 0 is the root.

This definition means informally that for every node w, if M_1 and M_2 agree on the interpretation from root to $last(w)$ then M_1 changes less than M_2 at the point from $last(w)$ to w.

Theorem 1. *$\preceq$ is a partial order relation, that is, $\preceq$ is a reflexive, transitive and anti-symmetric relation.*

The *minimal change* models are those models M such that there is no model M' such that $M' \prec M$ (which is equivalent to $M' \preceq M$ and $M' \not\equiv M$).

We can give a circumscription-like formula to compute the theorems, which are true in all minimal change models, in a similar way of [Kautz86]. Let $A(T)$ be an axiom set including predicate T and $A(\tau)$ be the set of sentences obtained by substituting τ for any occurrence of T in $A(T)$. $\tau = T$ stands for $\forall p \forall w[T(p, w) = \tau(p, w)]$ and $\tau \preceq T$ stands for the following expression.

$$
\begin{aligned}
&\forall p[T(p, 0) \equiv \tau(p, 0)] \wedge \\
&\forall w[
\end{aligned}
$$

$$[0 < w \;\wedge\; \forall w'[0 < w' < w \;\supset\; \forall p[T(p,w') \;\equiv\; \tau(p,w')]]] \;\supset$$
$$\forall p[[T(p,last(w)) \;\equiv\; T(p,w)] \;\supset\; [\tau(p,last(w)) \;\equiv\; \tau(p,w)]]$$
$$]$$

This expression means that T and τ are equivalent at 0 and for any w if T and τ are equivalent from 0 to $last(w)$ then changes of τ is less than changes of T. Now we give the circumscription-like formula.

$$A(T) \;\wedge\; \neg\exists\tau[A(\tau) \;\wedge\; \tau \prec T]$$

where $\tau \prec T$ stands for $\tau \preceq T$ and not $\tau \equiv T$.

The above formula is equivalent to the following formula.

$$A(T) \;\wedge\; \forall\tau[[A(\tau) \;\wedge\; \tau \preceq T] \;\supset\; \tau \equiv T]$$

The above formula means informally that if an arbitrary predicate τ which satisfies the conditions satisfied by T, and τ changes less than T at the earliest point which any difference occurs between T and τ, then T and τ are identical.

The relation between the above formula and minimal change models is as follows.

Theorem 2. *M is a model of $A(T) \;\wedge\; \neg\exists\tau[A(\tau) \;\wedge\; \tau \prec T]$ iff M is a minimal change model.*

The proofs of the above theorems are found in appendix A.

5. Examples

5.1 Tree-structured Inheritance System

We use the same example in the subsection 2.1. In this example, the root is ANIMAL, and node variables range over the set of classes, {ANIMAL, BIRD, PENGUIN, FISH, MAMMAL, BAT}, and property variables range over the set of properties, { FLY, SWIM}.

To define a partial order relation '<' over classes, we first define the following *parent-child relation* '⇒' over classes (this relation also denotes the function *last* such that $w_1 \Rightarrow w_2$ iff $last(w_2) = w_1$).

ANIMAL ⇒ BIRD

BIRD ⇒ PENGUIN

ANIMAL ⇒ FISH

ANIMAL ⇒ MAMMAL

MAMMAL ⇒ BAT

Then '<' is defined as transitive closure of the above relation, and if the pair of classes, (w_1, w_2), is not in the closure, it means $\neg(w_1 < w_2)$.

The information about properties is given as follows. Fig. 1 illustrates this information. In Fig. 1, a circle expresses a class and an arrow expresses an inheritance relation and a property is in a circle. If $T(P,W)$ is not true, the property in the circle is expressed as $\neg P$.

$\neg T$(FLY,ANIMAL)

T(FLY,BIRD)

$\neg T$(FLY,PENGUIN)

T(SWIM,FISH)

T(FLY,BAT)

Unique name axioms in classes and properties such as ANIMAL $\neq$ BIRD or FLY $\neq$ SWIM, are not shown.

Then the interpretations of predicate T by the minimal change models are as follows. Figs. 2 and 3 show these models.

$M_1[T]$ = { <FLY,BIRD>, <FLY,BAT>, <SWIM,FISH> }

$M_2[T]$ = { <FLY,BIRD>, <FLY,BAT>, <SWIM,ANIMAL>,
<SWIM,BIRD>, <SWIM,PENGUIN>, <SWIM,FISH>,
<SWIM,MAMMAL>, <SWIM,BAT> }

A common result is obtained for the two models illustrated in Fig. 4. Deduction of the common result by the formula in the previous section is found in appendix B.

It is easy to show that the other models change more than either of those models. For example, let

$M_3[T]$ = { <FLY,BIRD>, <FLY,MAMMAL>, <FLY,BAT>, <SWIM,FISH> }(see Fig.5).

Then $M_1 \prec M_3$, because at the point from ANIMAL to MAMMAL, M_3 changes more than M_1, that is, <FLY,ANIMAL> $\notin M_1[T]$(and $M_3[T]$) and <FLY,MAMMAL> $\notin M_1[T]$, but <FLY,MAMMAL> $\in M_3[T]$.

5.2 Temporal Projection

We use the same example in the subsection 2.2. In this example, the root is S_0, and action variables range over the set of actions, {LOAD, WAIT, SHOOT}, and node variables range over the set of situations,

{S_0, $RESULT$(LOAD,S_0), $RESULT$(WAIT,S_0), $RESULT$(SHOOT,S_0), ... },

and property variables range over the set of facts, {LOADED, ALIVE}.

To define a partial order relation '$<$' over situations, we first define the following *parent-child relation* '$\Rightarrow$' over classes (this relation also denotes the function *last* such that $w_1 \Rightarrow w_2$ iff $last(w_2) = w_1$).

$$\forall a \forall w[\ w \Rightarrow RESULT(a, w)]$$

Then '$<$' is defined as transitive closure of the above relation, and if the pair of classes, (w_1, w_2), is not in the closure, it means $\neg(w_1 < w_2)$.

The information about the state changes is expressed same as in subsection 2.2. Unique name axioms in actions and facts such as LOAD $\neq$ WAIT or LOADED $\neq$ ALIVE, and unique situation axioms such as $S_0 \neq RESULT(\text{LOAD}, S_0)$ are not shown.

Fig. 6 shows the facts to be true in all models of the above axioms.

The interpretations of predicate T by the minimal change models include the following results. Figs. 7 and 8 illustrate these models.

$M_1[T] \supset \{ <\text{ALIVE},S_0>, <\text{ALIVE},S_1>, <\text{LOADED},S_1>, <\text{ALIVE},S_2>, <\text{LOADED},S_2>, <\text{LOADED},S_3> \}$

$M_2[T] \supset \{ <\text{ALIVE},S_0>, <\text{LOADED},S_0>, <\text{ALIVE},S_1>, <\text{LOADED},S_1>, <\text{ALIVE},S_2>, <\text{LOADED},S_2>, <\text{LOADED},S_3> \}$

where $S_1 = RESULT(\text{LOAD},S_0)$ and $S_2 = RESULT(\text{WAIT},S_1)$ and $S_3 = RESULT(\text{SHOOT},S_2)$

A common result is obtained for the two models shown in Fig 9.

6. Related Research

In [Hanks87], it is pointed out that a problem in the temporal projection is to clear a preference criterion and to compute theorems common in all preferred models. This assertion can be also applied to tree-structured inheritance systems. We have given a preference criterion in these domains and a second-order formula to deduce theorems.

In this approach, [Kautz86] and [Shoham86] provide similar formalisms to ours in temporal projection. They formalize that people tend to think that facts persist as long as possible. [Shoham86] minimizes abnormality in the chronological order and [Kautz86] defines preferred models by later *Clip* of a fact. The difference between those formalism and ours is that they compare changes at the earliest point where any difference occurs between changes in models, while we compare them at the earliest point where any difference occurs between models. We explain the details by using the idea of [Kautz86].

In tree-structured multiple worlds, the idea of [Kautz86] can be translated into the following relation of models.

$M_1 \leq M_2$ iff

(1) M_1 and M_2 have the same domain.

(2) Every constant, function, and predicate symbol except T receives the same interpretation in M_1 and M_2.

(3) The following statement is true.

$$
\begin{array}{l}
\forall w[\\
\quad [0 < w \wedge\\
\quad\quad \forall w'[\\
\quad\quad\quad 0 < w' < w \supset\\
\quad\quad\quad \forall p[\\
\quad\quad\quad\quad [< p, last(w') > \in M_1[T] \equiv < p, w' > \in M_1[T]] \supset\\
\quad\quad\quad\quad [< p, last(w') > \in M_2[T] \equiv < p, w' > \in M_2[T]]\\
\quad\quad\quad]\\
\quad\quad]
\end{array}
$$

$$
\begin{array}{l}
\quad] \supset \\
\quad \forall p[\\
\qquad [< p, last(w) > \in M_2[T] \equiv < p, w > \in M_2[T]] \supset \\
\qquad [< p, last(w) > \in M_1[T] \equiv < p, w > \in M_1[T]] \\
\quad] \\
]
\end{array}
$$

In this definition, if M_1 is strictly better than M_2, M_1 and M_2 have identical changes up to some node, w', and at w', M_1 changes strictly less than M_2.

Then, in the above example of inheritance system, this criterion excludes the first model, since at point, from ANIMAL to FISH, M_2 changes strictly less than M_1 since M_1 has a change about the property, SWIM, but M_2 not. The detailed discussion is found in appendix C.

7. Conclusion

This paper formalizes two types of common sense reasoning, that is, tree-structured inheritance system and temporal projection. The main idea is that these types of reasoning can be regarded as meta-reasoning in tree-structured multiple worlds which selects preferred models among consistent models with the given information, and those preferred models change minimally in the direction of the tree. We believe that this formalism can be used as a clear specification of common sense reasoning system.

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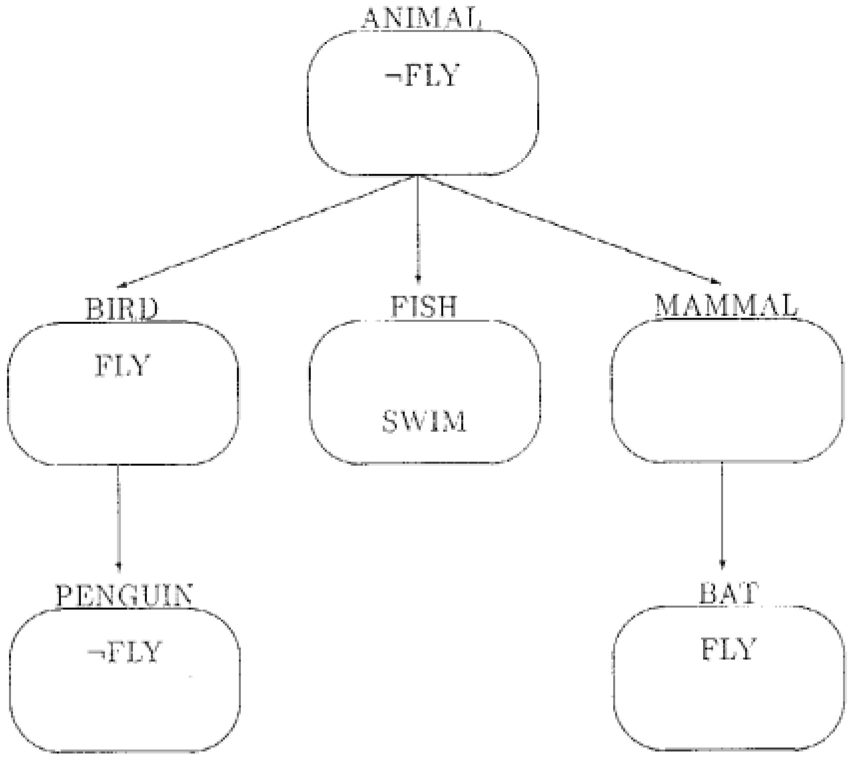

Fig. 1 Given Information

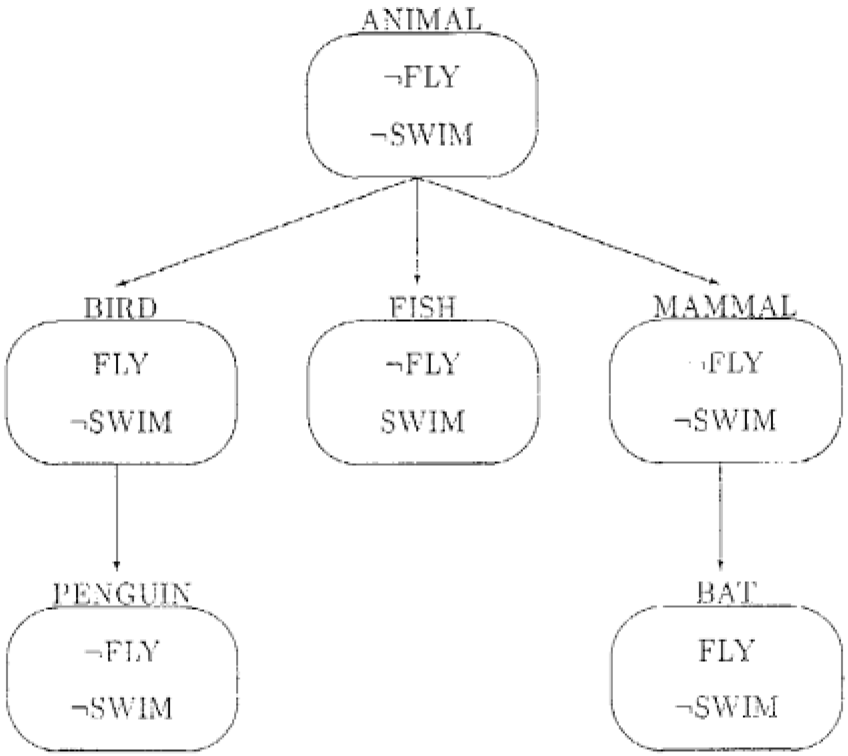

Fig. 2 Minimal Change Model M_1

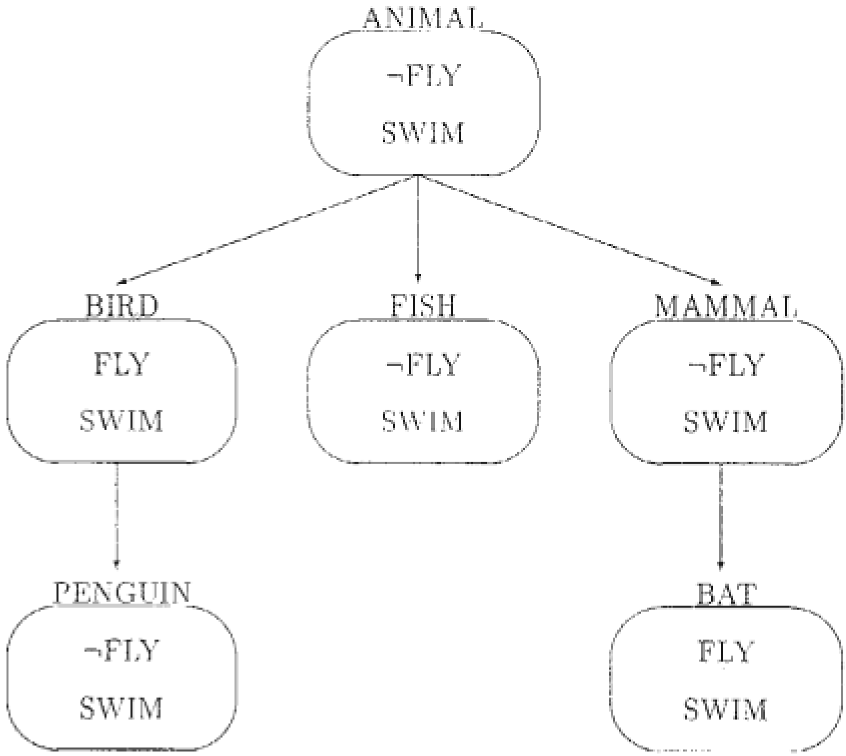

Fig. 3 Minimal Change Model M_2

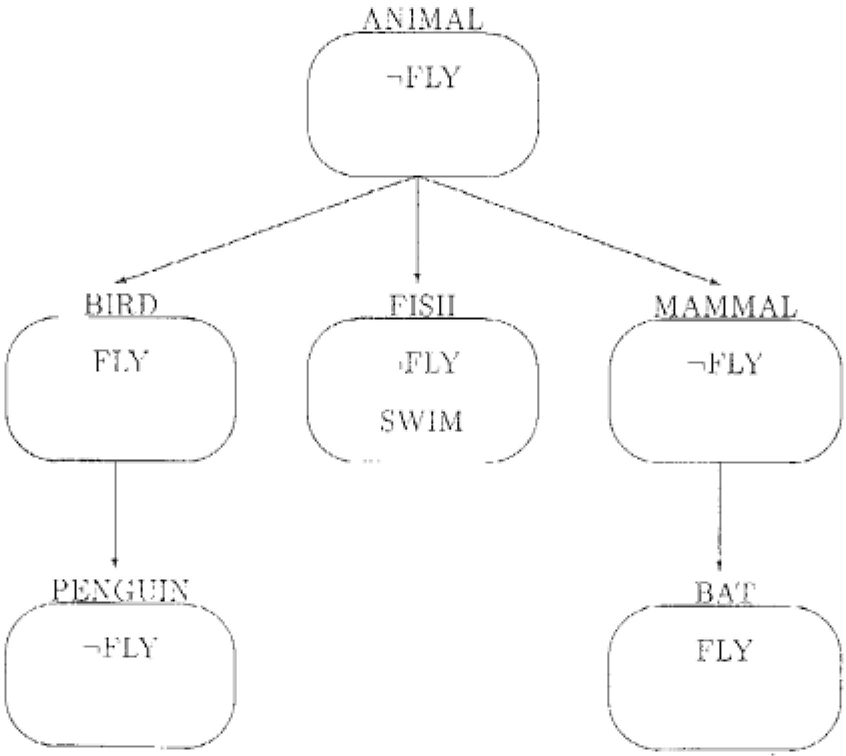

Fig. 4 Disjunction of Minimal Change Models

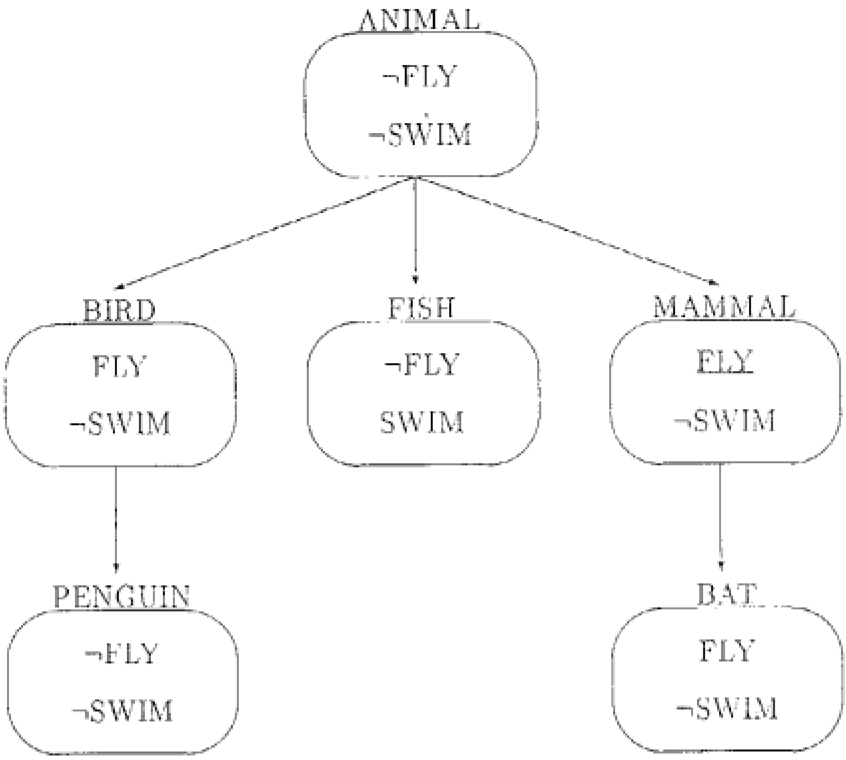

Fig. 5 Model M_3

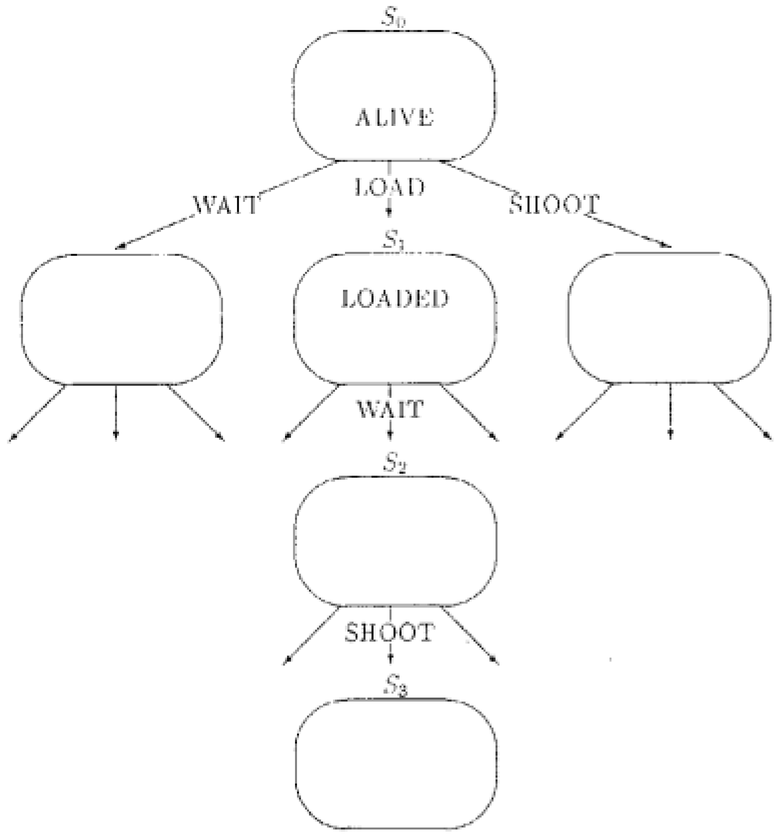

Fig. 6 Given Information

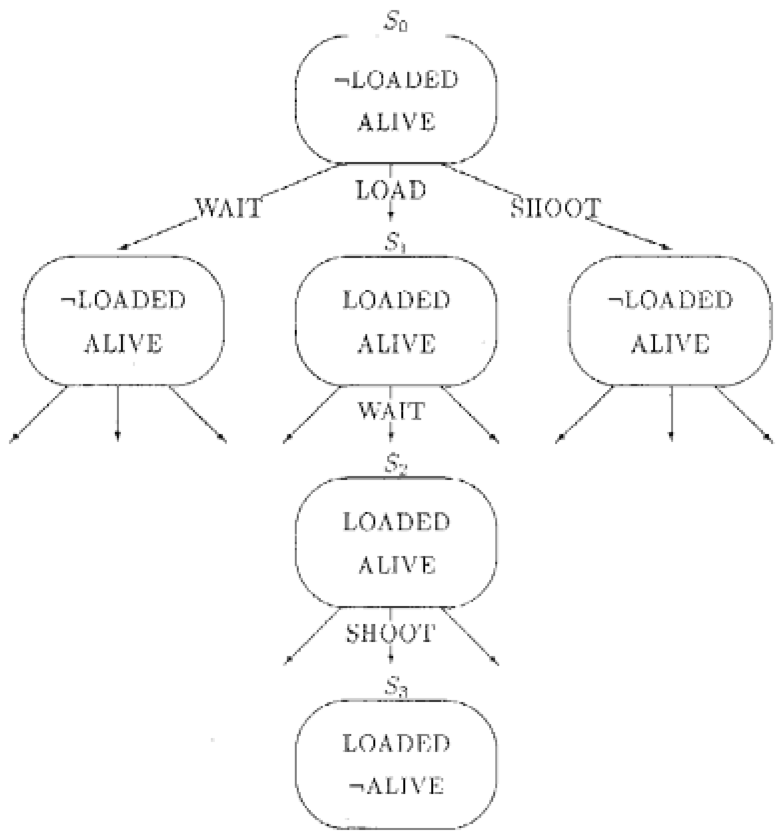

Fig. 7 Minimal Change Model M_1

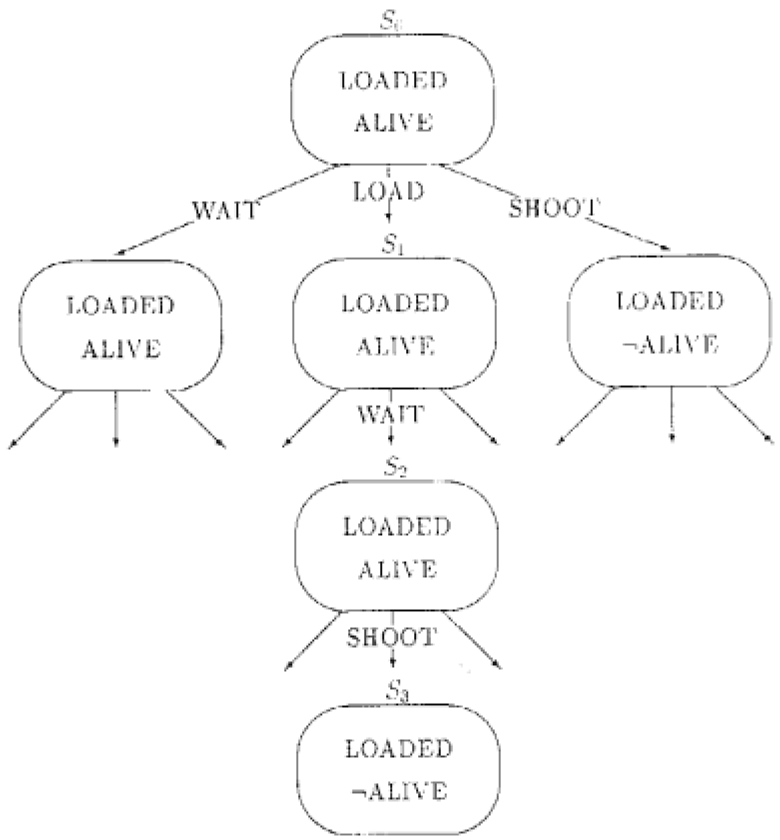

Fig. 8 Minimal Change Model M_2

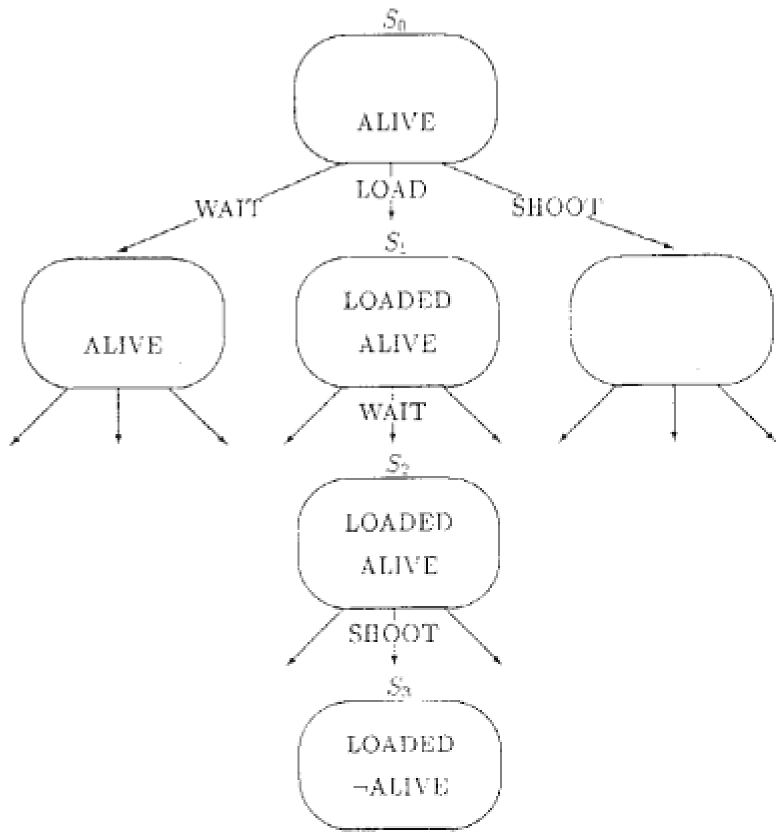

Fig. 9 Disjunction of Minimal Change Models

Appendix A. Proofs of Theorems

Theorem 1.

$\preceq$ is a partial order relation, that is, $\preceq$ is a reflexive, transitive and anti-symmetric relation.

Proof.

(1) reflexive property is trivial.

(2) transitive property

Suppose $M_1 \preceq M_2$ and $M_2 \preceq M_3$, but not $M_1 \preceq M_3$. Then,

There is a point W such that

$$[0 < W \wedge \forall w'[0 < w' < W \supset \forall p[< p, w' > \in M_1[T] \equiv < p, w' > \in M_3[T]]]] \wedge$$
$$\exists p[$$
$$[< p, last(W) > \in M_3[T] \equiv < p, W > \in M_3[T]] \wedge$$
$$[< p, last(W) > \in M_1[T] \not\equiv < p, W > \in M_1[T]]$$
$$].$$

Suppose there is a point W_1 and a property P such that

$0 < W_1 < W$,

$\forall w[0 < w < W_1 \supset \forall p[< p, w > \in M_1[T] \equiv < p, w > \in M_2[T]]]$ and

$< P, W_1 > \in M_1[T] \not\equiv < P, W_1 > \in M_2[T]$.

Then

$< P, W_1 > \in M_2[T] \not\equiv < P, W_1 > \in M_3[T]$,

because $< P, W_1 > \in M_1[T] \equiv < P, W_1 > \in M_3[T]$.

It means that there is a point W_2 and a property P' such that

$0 < W_2 \leq W_1$ and

$< P', W_2 > \in M_2[T] \not\equiv < P', W_2 > \in M_3[T]$.

Suppose $W_2 < W_1$ then

$< P', W_2 > \in M_1[T] \not\equiv < P', W_2 > \in M_3[T]$,

because $< P', W_2 > \in M_1[T] \equiv < P', W_2 > \in M_2[T]$.

It contradicts:

$\forall w'[0 < w' < W \supset \forall p[< p, w' > \in M_1[T] \equiv < p, w' > \in M_3[T]]]$.

Suppose $W_2 = W_1$ then

$[< P, last(W_1) > \in M_1[T] \equiv < P, W_1 > \in M_1[T]] \supset$

$[< P, last(W_1) > \in M_2[T] \equiv < P, W_1 > \in M_2[T]$ (because $M_1[T] \preceq M_2[T]$),

$< P, last(W_1) > \in M_1[T] \equiv < P, last(W_1) > \in M_2[T]$ and

$< P, W_1 > \in M_1[T] \not\equiv < P, W_1 > \in M_2[T]$.

Therefore

$$[< P, last(W_1) > \in M_2[T] \equiv < P, W_1 > \in M_2[T]]. \qquad (A.1)$$

And

$[< P, last(W_1) >\in M_2[T] \equiv< P, W_1 >\in M_2[T]] \supset$
$[< P, last(W_1) >\in M_3[T] \equiv< P, W_1 >\in M_3[T]]$ (because $M_2[T] \preceq M_3[T]$).
$< P, last(W_1) >\in M_2[T] = < P, last(W_1) >\in M_3[T]$ and
$< P, W_1 >\in M_2[T] \not\equiv < P, W_1 >\in M_3[T]$.

Therefore

$$[< P, last(W_1) >\in M_2[T] \not\equiv< P, W_1 >\in M_2[T]]. \quad (A.2)$$

(A.2) contradicts (A.1).

Therefore
$\forall w[0 < w < W \supset \forall p[< p, w >\in M_1[T] \equiv< p, w >\in M_2[T]]]$.

Therefore
$\forall p[[< p, last(W) >\in M_2[T] \equiv< p, W >\in M_2[T]] \supset$
$[< p, last(W) >\in M_1[T] \equiv< p, W >\in M_1[T]]]$,
because $M_1[T] \preceq M_2[T]$.

And since $\forall w[0 < w < W \supset \forall p[< p, w >\in M_1[T] \equiv< p, w >\in M_3[T]]]$,
$\forall w[0 < w < W \supset \forall p[< p, w >\in M_2[T] \equiv< p, w >\in M_3[T]]]$.

Therefore
$\forall p[[< p, last(W) >\in M_3[T] \equiv< p, W >\in M_3[T]] \supset$
$[< p, last(W) >\in M_2[T] \equiv< p, W >\in M_2[T]]]$,
because $M_2[T] \preceq M_3[T]$.

Therefore
$\forall p[[< p, last(W) >\in M_3[T] \equiv< p, W >\in M_3[T]] \supset$
$[< p, last(W) >\in M_1[T] \equiv< p, W >\in M_1[T]]]$.

It contradicts:
$\exists p[$
$[< p, last(W) >\in M_3[T] \equiv< p, W >\in M_3[T]]\wedge$
$[< p, last(W) >\in M_1[T] \not\equiv< p, W >\in M_1[T]]$
$]$.

(3) anti-symmetric property

Suppose $M_1 \preceq M_2$ and $M_2 \preceq M_1$ and $M_1 \not\equiv M_2$. Since $M_1 \not\equiv M_2$, there is a point W in a path where some difference occurs in $M_1[T]$ and $M_2[T]$. Then W is not the root because $\forall p[< p, 0 >\in M_1[T] \equiv < p, 0 >\in M_2[T]]$.

Since $\forall w[0 < w < W \supset \forall p[< p, w >\in M_1[T] \equiv < p, w >\in M_2[T]]]$,
$\forall p[< p, last(W) >\in M_1[T] \equiv < p, last(W) >\in M_2[T]]$.

And since the left-hand side of the second conjunct of each statement in $M_1 \preceq M_2$ and $M_2 \preceq M_1$ is true, the followings are true.

$\forall p[$
$[< p, last(W) >\in M_2[T] \equiv < p, W >\in M_2[T]] \supset$
$[< p, last(W) >\in M_1[T] \equiv < p, W >\in M_1[T]]$
$]$ and
$\forall p[$
$[< p, last(W) >\in M_1[T] \equiv < p, W >\in M_1[T]] \supset$
$[< p, last(W) >\in M_2[T] \equiv < p, W >\in M_2[T]]$

].

Therefore, $\forall p[< p, W > \in M_1[T] \equiv < p, W > \in M_2[T]]]$.
This contradicts the fact that W is the earliest point where any difference occurs in M_1 and M_2.

Theorem 2.

M is a model of $A(T) \wedge \neg\exists\tau[A(\tau) \wedge \tau \prec T]$ iff M is a minimal change model.

Proof.

($\Rightarrow$) Suppose $M \models A(T) \wedge \neg\exists\tau[A(\tau) \wedge \tau \prec T]$, but there is a model M' of $A(T)$ and $M' \prec M$.
Therefore

$$
\begin{array}{l}
\forall p[< p, 0 > \in M[T] \equiv < p, 0 > \in M'[T]] \wedge \\
\forall w[[0 < w \wedge \forall w'[0 < w' < w \supset \forall p[< p, w' > \in M[T] \equiv < p, w' > \in M'[T]]]] \supset \\
\quad \forall p[\\
\qquad [< p, last(w) > \in M[T] \equiv < p, w > \in M[T]] \supset \\
\qquad [< p, last(w) > \in M'[T] \equiv < p, w > \in M'[T]] \\
\quad] \\
] \wedge \\
\exists p \exists w[< p, w > \in M[T] \not\equiv < p, w > \in M'[T]].
\end{array}
$$

Let $M[\tau] = M'[T]$, where τ is a predicate constant which is not in $A(T)$, then since $M' \models A(T)$, $M \models A(\tau)$.
And since

$$
\begin{array}{l}
\forall p[< p, 0 > \in M[T] \equiv < p, 0 > \in M[\tau]] \wedge \\
\forall w[[0 < w \wedge \forall w'[0 < w' < w \supset \forall p[< p, w' > \in M[T] \equiv < p, w' > \in M[\tau]]]] \supset \\
\quad \forall p[\\
\qquad [< p, last(w) > \in M[T] \equiv < p, w > \in M[T]] \supset \\
\qquad [< p, last(w) > \in M[\tau] \equiv < p, w > \in M[\tau]] \\
\quad] \\
] \wedge \\
\exists p \exists w[< p, w > \in M[T] \not\equiv < p, w > \in M[\tau]]
\end{array}
$$

by substituting $M(\tau)$ for $M'(T)$ in the above statement,

$M \models \tau \prec T$.

It contradicts $M \models \neg\exists\tau[A(\tau) \wedge \tau \prec T]$.

($\Leftarrow$) Suppose M is a minimal change model and $M \vdash A(T) \wedge \exists\tau[A(\tau) \wedge \tau \prec T]$.
We take τ such that $A(\tau) \wedge \tau \prec T$.
Then since $M \models \tau \prec T$,

$$
\begin{array}{l}
\forall p[< p, 0 > \in M[T] \equiv < p, 0 > \in M[\tau]] \wedge \\
\forall w[[0 < w \wedge \forall w'[0 < w' < w \supset \forall p[< p, w' > \in M[T] \equiv < p, w' > \in M[\tau]]]] \supset \\
\quad \forall p[
\end{array}
$$

$$[< p, last(w) >\in M[T] \equiv < p, w >\subset M[T]] \supset$$
$$[< p, last(w) >\in M[\tau] \equiv < p, w >\subset M[\tau]]$$
$$]$$
$$] \wedge$$
$$\exists p \exists w [< p, w >\in M[T] \not\equiv < p, w >\in M[\tau]].$$

We can take M' such that it has the same domain as M and every function and every predicate other than T receives the same interpretation in M and M', and $M'[T]$ $= M[\tau]$. Since $M \models A(\tau)$, $M' \models A(T)$.
And since

$$\forall p[< p, 0 >\in M[T] \equiv < p, 0 >\in M'[T]] \wedge$$
$$\forall w[[0 < w \ \wedge \ \forall w'[0 < w' < w \supset \forall p[< p, w' >\in M[T] \equiv < p, w' >\in M'[T]]] \supset$$
$$\forall p[$$
$$[< p, last(w) >\in M[T] \ = < p, w >\in M[T]] \supset$$
$$[< p, last(w) >\in M'[T] \ \equiv < p, w >\in M'[T]]$$
$$]$$
$$] \wedge$$
$$\exists p \exists w [< p, w >\in M[T] \not\equiv < p, w >\in M'[T]]$$

by substituting $M'(T)$ for $M(\tau)$ in the above statement,

$M' \prec M$.

It contradicts the fact that M is a minimal change model.

Appendix B. Deduction of the Properties

Let $A(T)$ be the following axiom set.

$\neg T$(FLY,ANIMAL),

T(FLY,BIRD),

$\neg T$(FLY,PENGUIN),

T(SWIM,FISH),

T(FLY,BAT),

$\forall w_1 \forall w_2 [w_1 \Rightarrow w_2 \equiv [$

$[w_1 =$ ANIMAL $\wedge\ w_2 =$ BIRD$] \vee$

$[w_1 =$ BIRD $\wedge\ w_2 =$ PENGUIN$] \vee$

$[w_1 =$ ANIMAL $\wedge\ w_2 =$ FISH$] \vee$

$[w_1 =$ ANIMAL $\wedge\ w_2 =$ MAMMAL$] \vee$

$[w_1 =$ MAMMAL $\wedge\ w_2 =$ BAT$]$

$]]$,

$\forall w_1 \forall w_2 [w_1 < w_2 \equiv [w_1 \Rightarrow w_2 \vee \exists w_3 [w_1 < w_3 \wedge w_3 < w_2]]]$

and there are axioms stating that the root is ANIMAL, and node variables range over the set of classes, {ANIMAL, BIRD, PENGUIN, FISH, MAMMAL, BAT}, and property variables range over the set of properties, {FLY, SWIM}, and there are unique name axioms in classes and properties such as ANIMAL $\neq$ BIRD or FLY $\neq$ SWIM.

We will derive the common result in the minimal change models by the formula:

$$A(T) \ \wedge \ \forall\tau[[A(\tau) \ \wedge \ \tau \preceq T] \ \supset \ \tau \equiv T],$$

where $\tau \preceq T$ stands for:

$$\forall p[T(p,\text{ANIMAL}) \ \equiv \ \tau(p,\text{ANIMAL})] \wedge$$
$$\forall w[$$
$$[\text{ANIMAL} < w \ \wedge \ \forall w'[\text{ANIMAL} < w' < w \ \supset \ \forall p[T(p,w') \ \equiv \ \tau(p,w')]]] \ \supset$$
$$\forall p[[T(p,last(w)) \ \equiv \ T(p,w)] \ \supset \ [\tau(p,last(w)) \ \equiv \ \tau(p,w)]]$$
$$].$$

Let τ_1 be

$$\lambda p \lambda w[$$
$$[p = \text{FLY} \supset [w = \text{BIRD} \vee w = \text{BAT}]] \wedge$$
$$[p = \text{SWIM} \supset [w = \text{FISH}]]$$
$$].$$

$A(\tau_1)$ is true if $A(T)$ is true.
Therefore $[A(\tau_1) \wedge \tau_1 \preceq T] \supset [\tau_1 \equiv T]$ is simplified to $[\tau_1 \preceq T] \supset [\tau_1 \equiv T]$, if $A(T)$ is true.

Now we simplify the statement $[\tau_1 \preceq T] \supset [\tau_1 \equiv T]$ assuming $A(T)$.
At first, we simplify the first conjunct of $\tau_1 \preceq T$, that is,

$$\forall p[T(p,\text{ANIMAL}) \ \equiv \ \tau_1(p,\text{ANIMAL})],$$

which is equivalent to

$$[T(\text{FLY},\text{ANIMAL}) \equiv \tau_1(\text{FLY},\text{ANIMAL})] \wedge$$
$$[T(\text{SWIM},\text{ANIMAL}) \equiv \tau_1(\text{SWIM},\text{ANIMAL})].$$

Since $\neg T(\text{FLY},\text{ANIMAL})$ and $\neg\tau_1(\text{FLY},\text{ANIMAL})$ and $\neg\tau_1(\text{SWIM},\text{ANIMAL})$, the above is simplified to

$$\neg T(\text{SWIM},\text{ANIMAL}).$$

Then we simplify the second conjunct of $\tau_1 \preceq T$, that is,

$$\forall w[$$
$$[\text{ANIMAL} < w \ \wedge \ \forall w'[\text{ANIMAL} < w' < w \ \supset \ \forall p[T(p,w') \ \equiv \ \tau_1(p,w')]]] \ \supset$$
$$\forall p[[T(p,last(w)) \ \equiv \ T(p,w)] \ \supset \ [\tau_1(p,last(w)) \ \equiv \ \tau_1(p,w)]]$$
$$],$$

which is equivalent to

$$[\forall w'[\text{ANIMAL} < w' < \text{BIRD} \ \supset \ \forall p[T(p,w') \ \equiv \ \tau_1(p,w')]] \ \supset$$
$$\forall p[[T(p,\text{ANIMAL}) \ \equiv \ T(p,\text{BIRD})] \ \supset \ [\tau_1(p,\text{ANIMAL}) \ \equiv \ \tau_1(p,\text{BIRD})]]] \wedge$$
$$[\forall w'[\text{ANIMAL} < w' < \text{FISH} \ \supset \ \forall p[T(p,w') \ \equiv \ \tau_1(p,w')]] \ \supset$$
$$\forall p[[T(p,\text{ANIMAL})) \ \equiv \ T(p,\text{FISH})] \ \supset \ [\tau_1(p,\text{ANIMAL}) \ \equiv \ \tau_1(p,\text{FISH})]]] \wedge$$
$$[\forall w'[\text{ANIMAL} < w' < \text{MAMMAL} \ \supset \ \forall p[T(p,w') \ \equiv \ \tau_1(p,w')]] \ \supset$$

$$
\begin{aligned}
&\quad \forall p[[T(p, \text{ANIMAL}) \equiv T(p, \text{MAMMAL})] \supset [\tau_1(p, \text{ANIMAL}) \equiv \tau_1(p, \text{MAMMAL})]]] \wedge \\
&[\forall w'[\text{ANIMAL} < w' < \text{PENGUIN} \supset \forall p[T(p, w') \equiv \tau_1(p, w')]] \supset \\
&\quad \forall p[[T(p, \text{BIRD}) \equiv T(p, \text{PENGUIN})] \supset [\tau_1(p, \text{BIRD}) \equiv \tau_1(p, \text{PENGUIN})]]] \wedge \\
&[\forall w'[\text{ANIMAL} < w' < \text{BAT} \supset \forall p[T(p, w') \equiv \tau_1(p, w')]] \supset \\
&\quad \forall p[[T(p, \text{MAMMAL}) \equiv T(p, \text{BAT})] \supset [\tau_1(p, \text{MAMMAL}) \equiv \tau_1(p, \text{BAT})]]].
\end{aligned}
$$

The first conjunct is equivalent to:

$$
\begin{aligned}
&[[T(\text{FLY,ANIMAL}) \equiv T(\text{FLY,BIRD})] \supset \\
&\quad [\tau_1(\text{FLY,ANIMAL}) \equiv \tau_1(\text{FLY,BIRD})]] \wedge \\
&[[T(\text{SWIM,ANIMAL}) \equiv T(\text{SWIM,BIRD})] \supset \\
&\quad [\tau_1(\text{SWIM,ANIMAL}) \equiv \tau_1(\text{SWIM,BIRD})]],
\end{aligned}
$$

which is true because $T(\text{FLY,ANIMAL}) \not\equiv T(\text{FLY,BIRD})$ and $\tau_1(\text{SWIM,ANIMAL}) \equiv \tau_1(\text{SWIM,BIRD})$.

The second conjunct is equivalent to:

$$
\begin{aligned}
&[[T(\text{FLY,ANIMAL}) \equiv T(\text{FLY,FISH})] \supset \\
&\quad [\tau_1(\text{FLY,ANIMAL}) \equiv \tau_1(\text{FLY,FISH})]] \wedge \\
&[[T(\text{SWIM,ANIMAL}) \equiv T(\text{SWIM,FISH})] \supset \\
&\quad [\tau_1(\text{SWIM,ANIMAL}) \equiv \tau_1(\text{SWIM,FISH})]],
\end{aligned}
$$

which is equivalent to $\neg T(\text{SWIM,ANIMAL})$ because
$\tau_1(\text{FLY,ANIMAL}) \equiv \tau_1(\text{FLY,FISH})$, $T(\text{SWIM,FISH})$ and
$\tau_1(\text{SWIM,ANIMAL}) \not\equiv \tau_1(\text{SWIM,FISH})$.

The third conjunct is equivalent to:

$$
\begin{aligned}
&[[T(\text{FLY,ANIMAL}) \equiv T(\text{FLY,MAMMAL})] \supset \\
&\quad [\tau_1(\text{FLY,ANIMAL}) \equiv \tau_1(\text{FLY,MAMMAL})]] \wedge \\
&[[T(\text{SWIM,ANIMAL}) \equiv T(\text{SWIM,MAMMAL})] \supset \\
&\quad [\tau_1(\text{SWIM,ANIMAL}) \equiv \tau_1(\text{SWIM,MAMMAL})]],
\end{aligned}
$$

which is true because $\tau_1(\text{FLY,ANIMAL}) \equiv \tau_1(\text{FLY,MAMMAL})$
and $\tau_1(\text{SWIM,ANIMAL}) \equiv \tau_1(\text{SWIM,MAMMAL})$.

The fourth conjunct is equivalent to:

$$
\begin{aligned}
&[[T(\text{FLY,BIRD}) \equiv \tau_1(\text{FLY,BIRD})] \wedge [T(\text{SWIM,BIRD}) \equiv \tau_1(\text{SWIM,BIRD})]] \supset \\
&[[[T(\text{FLY,BIRD}) \equiv T(\text{FLY,PENGUIN})] \supset \\
&\quad [\tau_1(\text{FLY,BIRD}) \equiv \tau_1(\text{FLY,PENGUIN})]] \wedge \\
&[[T(\text{SWIM,BIRD}) \equiv T(\text{SWIM,PENGUIN})] \supset \\
&\quad [\tau_1(\text{SWIM,BIRD}) \equiv \tau_1(\text{SWIM,PENGUIN})]]],
\end{aligned}
$$

which is true because the right-hand side is true.

The fifth conjunct is equivalent to:

$$
\begin{aligned}
&[[T(\text{FLY,MAMMAL}) \equiv \tau_1(\text{FLY,MAMMAL})] \wedge \\
&\quad [T(\text{SWIM,MAMMAL}) \equiv \tau_1(\text{SWIM,MAMMAL})]] \supset
\end{aligned}
$$

$$[[[T(\text{FLY,MAMMAL}) \equiv T(\text{FLY,BAT})] \supset [\tau_1(\text{FLY,MAMMAL}) \equiv \tau_1(\text{FLY,BAT})]] \wedge [[T(\text{SWIM,MAMMAL}) \equiv T(\text{SWIM,BAT})] \supset [\tau_1(\text{SWIM,MAMMAL}) \equiv \tau_1(\text{SWIM,BAT})]]],$$

which is equivalent to:

$$[\neg T(\text{FLY,MAMMAL}) \wedge \neg T(\text{SWIM,MAMMAL})] \supset \neg T(\text{FLY,MAMMAL})$$

which is true.

Therefore the above statement is simplified to:

$$\neg T(\text{SWIM,ANIMAL}) \supset [\tau_1 \equiv T], \quad (B.1)$$

assuming $A(T)$.

Let τ_2 be

$$\lambda p \lambda w[\; [p = \text{FLY} \supset [w = \text{BIRD} \vee w = \text{BAT}]] \wedge [p = \text{SWIM} \supset [w = \text{ANIMAL} \vee w = \text{BIRD} \vee w = \text{FISH} \vee w = \text{MAMMAL} \vee w = \text{PENGUIN} \vee w = \text{BAT}] \;] \;].$$

$A(\tau_2)$ is true if $A(T)$ is true.
Therefore $[A(\tau_2) \wedge \tau_2 \preceq T] \supset [\tau_2 \equiv T]$ is simplified to $[\tau_2 \preceq T] \supset [\tau_2 \equiv T]$, if $A(T)$ is true.

Now we simplify the statement $[\tau_2 \preceq T] \supset [\tau_2 \equiv T]$ assuming $A(T)$.
At first, we simplify the first conjunct of $\tau_2 \preceq T$, that is,

$$\forall p[T(p,\text{ANIMAL}) \equiv \tau_2(p,\text{ANIMAL})],$$

which is equivalent to

$$[T(\text{FLY,ANIMAL}) \equiv \tau_2(\text{FLY,ANIMAL})] \wedge [T(\text{SWIM,ANIMAL}) \equiv \tau_2(\text{SWIM,ANIMAL})].$$

Since $\neg T(\text{FLY,ANIMAL})$ and $\neg\tau_2(\text{FLY,ANIMAL})$ and $\tau_2(\text{SWIM,ANIMAL})$, the above is simplified to

$$T(\text{SWIM,ANIMAL}).$$

Then we simplify the second conjunct of $\tau_2 \preceq T$, that is,

$$\forall w[\; [\text{ANIMAL} < w \wedge \forall w'[\text{ANIMAL} < w' < w \supset \forall p[T(p,w') \equiv \tau_2(p,w')]]] \supset \forall p[[T(p,last(w)) \equiv T(p,w)] \supset [\tau_2(p,last(w)) \equiv \tau_2(p,w)]] \;],$$

which is equivalent to

$$[\forall w'[\text{ANIMAL} < w' < \text{BIRD} \supset \forall p[T(p,w') \equiv \tau_2(p,w')]] \supset$$

$$
\begin{aligned}
&\quad \forall p[[T(p,\mathrm{ANIMAL}) \equiv T(p,\mathrm{BIRD})] \supset [\tau_2(p,\mathrm{ANIMAL}) \equiv \tau_2(p,\mathrm{BIRD})]]] \wedge \\
&[\forall w'[\mathrm{ANIMAL} < w' < \mathrm{FISH} \supset \forall p[T(p,w') \equiv \tau_2(p,w')]] \supset \\
&\quad \forall p[[T(p,\mathrm{ANIMAL})) \equiv T(p,\mathrm{FISH})] \supset [\tau_2(p,\mathrm{ANIMAL}) \equiv \tau_2(p,\mathrm{FISH})]]] \wedge \\
&[\forall w'[\mathrm{ANIMAL} < w' < \mathrm{MAMMAL} \supset \forall p[T(p,w') \equiv \tau_2(p,w')]] \supset \\
&\quad \forall p[[T(p,\mathrm{ANIMAL}) \equiv T(p,\mathrm{MAMMAL})] \supset [\tau_2(p,\mathrm{ANIMAL}) \equiv \tau_2(p,\mathrm{MAMMAL})]]] \wedge \\
&[\forall w'[\mathrm{ANIMAL} < w' < \mathrm{PENGUIN} \supset \forall p[T(p,w') \equiv \tau_2(p,w')]] \supset \\
&\quad \forall p[[T(p,\mathrm{BIRD}) \equiv T(p,\mathrm{PENGUIN})] \supset [\tau_2(p,\mathrm{BIRD}) \equiv \tau_2(p,\mathrm{PENGUIN})]]] \wedge \\
&[\forall w'[\mathrm{ANIMAL} < w' < \mathrm{BAT} \supset \forall p[T(p,w') \equiv \tau_2(p,w')]] \supset \\
&\quad \forall p[[T(p,\mathrm{MAMMAL}) \equiv T(p,\mathrm{BAT})] \supset [\tau_2(p,\mathrm{MAMMAL}) \equiv \tau_2(p,\mathrm{BAT})]]].
\end{aligned}
$$

The first conjunct is equivalent to:

$$
\begin{aligned}
&[[T(\mathrm{FLY,ANIMAL}) \equiv T(\mathrm{FLY,BIRD})] \supset \\
&\quad [\tau_2(\mathrm{FLY,ANIMAL}) \equiv \tau_2(\mathrm{FLY,BIRD})]] \wedge \\
&[[T(\mathrm{SWIM,ANIMAL}) \equiv T(\mathrm{SWIM,BIRD})] \supset \\
&\quad [\tau_2(\mathrm{SWIM,ANIMAL}) \equiv \tau_2(\mathrm{SWIM,BIRD})]],
\end{aligned}
$$

which is true because $T(\mathrm{FLY,ANIMAL}) \not\equiv T(\mathrm{FLY,BIRD})$ and $\tau_2(\mathrm{SWIM,ANIMAL}) \equiv \tau_2(\mathrm{SWIM,BIRD})$.

The second conjunct is equivalent to:

$$
\begin{aligned}
&[[T(\mathrm{FLY,ANIMAL}) \equiv T(\mathrm{FLY,FISH})] \supset \\
&\quad [\tau_2(\mathrm{FLY,ANIMAL}) \equiv \tau_2(\mathrm{FLY,FISH})]] \wedge \\
&[[T(\mathrm{SWIM,ANIMAL}) \equiv T(\mathrm{SWIM,FISH})] \supset \\
&\quad [\tau_2(\mathrm{SWIM,ANIMAL}) \equiv \tau_2(\mathrm{SWIM,FISH})]],
\end{aligned}
$$

which is true because $\tau_2(\mathrm{FLY,ANIMAL}) \equiv \tau_2(\mathrm{FLY,FISH})$ and $\tau_2(\mathrm{SWIM,ANIMAL}) \equiv \tau_2(\mathrm{SWIM,FISH})$.

The third conjunct is equivalent to:

$$
\begin{aligned}
&[[T(\mathrm{FLY,ANIMAL}) \equiv T(\mathrm{FLY,MAMMAL})] \supset \\
&\quad [\tau_2(\mathrm{FLY,ANIMAL}) \equiv \tau_2(\mathrm{FLY,MAMMAL})]] \wedge \\
&[[T(\mathrm{SWIM,ANIMAL}) \equiv T(\mathrm{SWIM,MAMMAL})] \supset \\
&\quad [\tau_2(\mathrm{SWIM,ANIMAL}) \equiv \tau_2(\mathrm{SWIM,MAMMAL})]],
\end{aligned}
$$

which is true because $\tau_2(\mathrm{FLY,ANIMAL}) \equiv \tau_2(\mathrm{FLY,MAMMAL})$ and $\tau_2(\mathrm{SWIM,ANIMAL}) \equiv \tau_2(\mathrm{SWIM,MAMMAL})$.

The fourth conjunct is equivalent to:

$$
\begin{aligned}
&[[T(\mathrm{FLY,BIRD}) \equiv \tau_2(\mathrm{FLY,BIRD})] \wedge [T(\mathrm{SWIM,BIRD}) \equiv \tau_2(\mathrm{SWIM,BIRD})]] \supset \\
&[[[T(\mathrm{FLY,BIRD}) \equiv T(\mathrm{FLY,PENGUIN})] \supset \\
&\quad [\tau_2(\mathrm{FLY,BIRD}) \equiv \tau_2(\mathrm{FLY,PENGUIN})]] \wedge \\
&[[T(\mathrm{SWIM,BIRD}) \equiv T(\mathrm{SWIM,PENGUIN})] \supset \\
&\quad [\tau_2(\mathrm{SWIM,BIRD}) \equiv \tau_2(\mathrm{SWIM,PENGUIN})]]],
\end{aligned}
$$

which is true because the right-hand side is true.

The fifth conjunct is equivalent to:

$$[[T(\text{FLY,MAMMAL}) \equiv \tau_2(\text{FLY,MAMMAL})] \wedge$$
$$[T(\text{SWIM,MAMMAL}) \equiv \tau_2(\text{SWIM,MAMMAL})]] \supset$$
$$[[[T(\text{FLY,MAMMAL}) \equiv T(\text{FLY,BAT})] \supset$$
$$[\tau_2(\text{FLY,MAMMAL}) \equiv \tau_2(\text{FLY,BAT})]] \wedge$$
$$[[T(\text{SWIM,MAMMAL}) \equiv T(\text{SWIM,BAT})] \supset$$
$$[\tau_2(\text{SWIM,MAMMAL}) \equiv \tau_2(\text{SWIM,BAT})]]],$$

which is equivalent to:

$$[\neg T(\text{FLY,MAMMAL}) \wedge T(\text{SWIM,MAMMAL})] \supset \neg T(\text{FLY,MAMMAL}),$$

which is true.

Therefore the above statement is simplified to:

$$T(\text{SWIM,ANIMAL}) \supset [\tau_2 \equiv T], \tag{B.2}$$

assuming $A(T)$.

From (B.1) and (B.2),

$$A(T) \wedge \neg\exists\tau[A(\tau) \wedge \tau < T] \vdash [T \equiv \tau_1] \vee [T \equiv \tau_2],$$

which means

$$A(T) \wedge \neg\exists\tau[A(\tau) \wedge \tau < T] \vdash \neg T(\text{FLY,FISH}) \wedge \neg T(\text{FLY,MAMMAL}).$$

Appendix C. Comparison with [Kautz86]

The original definition of a partial order relation, $\leq$, over models is the following.

$M_1 \leq M_2$ iff

(1) M_1 and M_2 have the same domain.

(2) Every constant, function, and predicate symbol except $Hold$ and $Clip$ receives the same interpretation in M_1 and M_2.

(3) The following statement is true:

$$\forall f \forall t[$$
$$< f,t > \in M_1[Clip] \supset [$$
$$< f,t > \in M_2[Clip] \vee$$
$$\exists f' \exists t'[$$
$$< f',t' > \in M_2[Clip] \wedge$$
$$< f',t' > \notin M_1[Clip] \wedge$$
$$< t',t > \in M_1[<]$$
$$]$$
$$]$$
$$]$$

where $Hold(f,t) \supset [Hold(f,t+1) \oplus Clip(f,t+1)]$.

The above statement is equivalent to:

$\forall t[$
$\quad \forall t'[$
$\quad\quad < t', t > \in M_1[<] \supset$
$\quad\quad \forall f[$
$\quad\quad\quad < f, t' > \notin M_1[Clip] \supset$
$\quad\quad\quad < f, t' > \notin M_2[Clip]$
$\quad\quad]$
$\quad] \supset$
$\quad \forall f[$
$\quad\quad < f, t > \notin M_2[Clip] \supset$
$\quad\quad < f, t > \notin M_1[Clip]$
$\quad]$
$].$

This definition is different from our formalism in the points below.

(1) t is totally ordered while w in our formalism is partially ordered.

(2) Since f in $Hold(f, t)$ is always a constant in his paper, the fomula, $Hold(f, t) \supset [Hold(f, t+1) \oplus Clip(f, t+1)]$, can only be used for true facts and cannot be used for false facts. It means that his formalism expresses only the persistence of true facts.

(3) Once $Hold$ becomes false, we cannot apply the above formula after that. Therefore we cannot tell whether its negation will persist or not.

So we modify his formalism to express the persistence of fact (including the persistence of its negation) in a tree-structured multiple world.

To express persistence of the negation, we modify the definition of $Clip$ to the following:

$Hold(f, t) \equiv Clip(f, t+1) \oplus Hold(f, t+1)$,

which is equivalent to: $Clip(f, t+1) \equiv Hold(f, t) \not\equiv Hold(f, t+1)$.

To express the above formula in a tree-structured multiple world, we modify the definition of t to w which is partially ordered. Since $t+1$ is no longer a function, we use $last(w)$ in stead of $t-1$.

Then

$Clip(f, t) \equiv Hold(f, t-1) \not\equiv Hold(f, t)$

becomes:

$Clip(f, w) \equiv Hold(f, last(w)) \not\equiv Hold(f, w)$.

And since we cannot define $Clip$ at 0, w must be more than 0.

Therefore the above statement becomes:

$$
\begin{array}{l}
\forall w[\\
\quad [< 0, w > \in M_1[<] \wedge \\
\quad\quad \forall w'[\\
\quad\quad\quad [< 0, w' > \in M_1[<] \wedge < w', w > \in M_1[<]] \supset \\
\quad\quad\quad \forall f[\\
\quad\quad\quad\quad [< f, last(w') > \in M_1[Hold] \equiv < f, w' > \in M_1[Hold]] \supset \\
\quad\quad\quad\quad [< f, last(w') > \in M_2[Hold] \equiv < f, w' > \in M_2[Hold]] \\
\quad\quad\quad] \\
\quad\quad] \\
\quad] \supset \\
\quad \forall f[\\
\quad\quad [< f, last(w) > \in M_2[Hold] \equiv < f, w > \in M_2[Hold]] \supset \\
\quad\quad [< f, last(w) > \in M_1[Hold] \equiv < f, w > \in M_1[Hold]] \\
\quad] \\
],
\end{array}
$$

which is found in section 6.

Now we can compare his formalism with ours.
Let $A(T)$ be the following axiom set.

$$
\begin{array}{l}
\neg T(\text{FLY},\text{ANIMAL}), \\
T(\text{FLY},\text{BIRD}), \\
\neg T(\text{FLY},\text{PENGUIN}), \\
T(\text{SWIM},\text{FISH}), \\
T(\text{FLY},\text{BAT}), \\
\forall w_1 \forall w_2 [w_1 \Rightarrow w_2 \equiv [\\
\quad\quad [w_1 = \text{ANIMAL} \wedge w_2 = \text{BIRD}] \vee \\
\quad\quad [w_1 = \text{BIRD} \wedge w_2 = \text{PENGUIN}] \vee \\
\quad\quad [w_1 = \text{ANIMAL} \wedge w_2 = \text{FISH}] \vee \\
\quad\quad [w_1 = \text{ANIMAL} \wedge w_2 = \text{MAMMAL}] \vee \\
\quad\quad [w_1 = \text{MAMMAL} \wedge w_2 = \text{BAT}] \\
]], \\
\forall w_1 \forall w_2 [w_1 < w_2 \equiv [w_1 \Rightarrow w_2 \vee \exists w_3 [w_1 < w_3 \wedge w_3 < w_2]]]
\end{array}
$$

and there are axioms stating that the root is ANIMAL, and node variables range over the set of classes, {ANIMAL, BIRD, PENGUIN, FISH, MAMMAL, BAT}, and property variables range over the set of properties, {FLY, SWIM},

and there are unique name axioms in classes and properties such as ANIMAL $\neq$ BIRD or FLY $\neq$ SWIM.

We will derive the common result in the most persistent models by the formula:

$A(T) \wedge \forall\tau[[A(\tau) \wedge \tau \leq T] \supset T \leq \tau]$,

where $\tau \leq T$ stands for:

$\forall w[$

$[\mathrm{ANIMAL} < w \wedge \forall w'[\mathrm{ANIMAL} < w' < w \supset$

$\forall p[[\tau(p, last(w')) \equiv \tau(p, w')] \supset [T(p, last(w')) \equiv T(p, w')]]$

$] \supset$

$\forall p[[T(p, last(w)) \equiv T(p, w)] \supset [\tau(p, last(w)) \equiv \tau(p, w)]]$

$]$.

Let τ be

$\lambda p \lambda w[$

$[p = \mathrm{FLY} \supset [w = \mathrm{BIRD} \vee w = \mathrm{BAT}]] \wedge$

$[p = \mathrm{SWIM} \supset$

$[w = \mathrm{ANIMAL} \vee w = \mathrm{BIRD} \vee w = \mathrm{FISH} \vee$

$w = \mathrm{MAMMAL} \vee w = \mathrm{PENGUIN} \vee w = \mathrm{BAT}]$

$]$

$]$.

$A(\tau)$ is true if $A(T)$ is true.
Therefore $[A(\tau) \wedge \tau \leq T] \supset [T \leq \tau]$ is simplified to $[\tau \leq T] \supset [T \leq \tau]$, if $A(T)$ is true.

Now we simplify the statement $[\tau \leq T] \supset [T \leq \tau]$ assuming $A(T)$.

$\tau \leq T$ is equivalent to:

$[\forall w'[\mathrm{ANIMAL} < w' < \mathrm{BIRD} \supset$

$\forall p[[\tau(p, last(w')) \equiv \tau(p, w')] \supset [T(p, last(w')) \equiv T(p, w')]]] \supset$

$\forall p[[T(p, \mathrm{ANIMAL}) \equiv T(p, \mathrm{BIRD})] \supset$

$[\tau(p, \mathrm{ANIMAL}) \equiv \tau(p, \mathrm{BIRD})]]] \wedge$

$[\forall w'[\mathrm{ANIMAL} < w' < \mathrm{FISH} \supset$

$\forall p[[\tau(p, last(w')) \equiv \tau(p, w')] \supset [T(p, last(w')) \equiv T(p, w')]]] \supset$

$\forall p[[T(p, \mathrm{ANIMAL})) \equiv T(p, \mathrm{FISH})] \supset$

$[\tau(p, \mathrm{ANIMAL}) \equiv \tau(p, \mathrm{FISH})]]] \wedge$

$[\forall w'[\mathrm{ANIMAL} < w' < \mathrm{MAMMAL} \supset$

$\forall p[[\tau(p, last(w')) \equiv \tau(p, w')] \supset [T(p, last(w')) \equiv T(p, w')]]] \supset$

$\forall p[[T(p, \mathrm{ANIMAL}) \equiv T(p, \mathrm{MAMMAL})] \supset$

$[\tau(p, \mathrm{ANIMAL}) \equiv \tau(p, \mathrm{MAMMAL})]]] \wedge$

$[\forall w'[\mathrm{ANIMAL} < w' < \mathrm{PENGUIN} \supset$

$\forall p[[\tau(p, last(w')) \equiv \tau(p, w')] \supset [T(p, last(w')) \equiv T(p, w')]]] \supset$

$\forall p[[T(p, \mathrm{BIRD}) \equiv T(p, \mathrm{PENGUIN})] \supset$

$[\tau(p, \mathrm{BIRD}) \equiv \tau(p, \mathrm{PENGUIN})]]] \wedge$

$[\forall w'[\mathrm{ANIMAL} < w' < \mathrm{BAT} \supset$

$\forall p[[\tau(p, last(w')) \equiv \tau(p, w')] \supset [T(p, last(w')) \equiv T(p, w')]]] \supset$

$\forall p[[T(p, \mathrm{MAMMAL}) \equiv T(p, \mathrm{BAT})] \supset$

$[\tau(p, \mathrm{MAMMAL}) \equiv \tau(p, \mathrm{BAT})]]]$.

The first conjunct is equivalent to:

$[[T(\mathrm{FLY},\mathrm{ANIMAL}) \equiv T(\mathrm{FLY},\mathrm{BIRD})] \supset$
$\quad [\tau(\mathrm{FLY},\mathrm{ANIMAL}) \equiv \tau(\mathrm{FLY},\mathrm{BIRD})]] \wedge$
$[[T(\mathrm{SWIM},\mathrm{ANIMAL}) \equiv T(\mathrm{SWIM},\mathrm{BIRD})] \supset$
$\quad [\tau(\mathrm{SWIM},\mathrm{ANIMAL}) \equiv \tau(\mathrm{SWIM},\mathrm{BIRD})]],$

which is true because $T(\mathrm{FLY},\mathrm{ANIMAL}) \not\equiv T(\mathrm{FLY},\mathrm{BIRD})$ and $\tau(\mathrm{SWIM},\mathrm{ANIMAL}) \equiv \tau(\mathrm{SWIM},\mathrm{BIRD})$.

The second conjunct is equivalent to:

$[[T(\mathrm{FLY},\mathrm{ANIMAL}) \equiv T(\mathrm{FLY},\mathrm{FISH})] \supset$
$\quad [\tau(\mathrm{FLY},\mathrm{ANIMAL}) \equiv \tau(\mathrm{FLY},\mathrm{FISH})]] \wedge$
$[[T(\mathrm{SWIM},\mathrm{ANIMAL}) \equiv T(\mathrm{SWIM},\mathrm{FISH})] \supset$
$\quad [\tau(\mathrm{SWIM},\mathrm{ANIMAL}) \equiv \tau(\mathrm{SWIM},\mathrm{FISH})]],$

which is true because $\tau(\mathrm{FLY},\mathrm{ANIMAL}) \equiv \tau(\mathrm{FLY},\mathrm{FISH})$ and $\tau(\mathrm{SWIM},\mathrm{ANIMAL}) \equiv \tau(\mathrm{SWIM},\mathrm{FISH})$.

The third conjunct is equivalent to:

$[[T(\mathrm{FLY},\mathrm{ANIMAL}) \equiv T(\mathrm{FLY},\mathrm{MAMMAL})] \supset$
$\quad [\tau(\mathrm{FLY},\mathrm{ANIMAL}) \equiv \tau(\mathrm{FLY},\mathrm{MAMMAL})]] \wedge$
$[[T(\mathrm{SWIM},\mathrm{ANIMAL}) \equiv T(\mathrm{SWIM},\mathrm{MAMMAL})] \supset$
$\quad [\tau(\mathrm{SWIM},\mathrm{ANIMAL}) \equiv \tau(\mathrm{SWIM},\mathrm{MAMMAL})]],$

which is true because $\tau(\mathrm{FLY},\mathrm{ANIMAL}) \equiv \tau(\mathrm{FLY},\mathrm{MAMMAL})$
and $\tau(\mathrm{SWIM},\mathrm{ANIMAL}) \equiv \tau(\mathrm{SWIM},\mathrm{MAMMAL})$.

The fourth conjunct is equivalent to:

$[[[\tau(\mathrm{FLY},\mathrm{ANIMAL}) \equiv \tau(\mathrm{FLY},\mathrm{BIRD})] \supset$
$\quad [T(\mathrm{FLY},\mathrm{ANIMAL}) \equiv T(\mathrm{FLY},\mathrm{BIRD})]] \wedge$
$[[\tau(\mathrm{SWIM},\mathrm{ANIMAL}) \equiv \tau(\mathrm{SWIM},\mathrm{BIRD})] \supset$
$\quad [T(\mathrm{SWIM},\mathrm{ANIMAL}) \equiv T(\mathrm{SWIM},\mathrm{BIRD})]]] \supset$
$[[[T(\mathrm{FLY},\mathrm{BIRD}) \equiv T(\mathrm{FLY},\mathrm{PENGUIN})] \supset$
$\quad [\tau(\mathrm{FLY},\mathrm{BIRD}) \equiv \tau(\mathrm{FLY},\mathrm{PENGUIN})]] \wedge$
$[[T(\mathrm{SWIM},\mathrm{BIRD}) \equiv T(\mathrm{SWIM},\mathrm{PENGUIN})] \supset$
$\quad [\tau(\mathrm{SWIM},\mathrm{BIRD}) \equiv \tau(\mathrm{SWIM},\mathrm{PENGUIN})]]],$

which is true because the right-hand side is true.

The fifth conjunct is equivalent to:

$[[[\tau(\mathrm{FLY},\mathrm{ANIMAL}) \equiv \tau(\mathrm{FLY},\mathrm{MAMMAL})] \supset$
$\quad [T(\mathrm{FLY},\mathrm{ANIMAL}) \equiv T(\mathrm{FLY},\mathrm{MAMMAL})]] \wedge$
$[[\tau(\mathrm{SWIM},\mathrm{ANIMAL}) \equiv \tau(\mathrm{SWIM},\mathrm{MAMAMAL})] \supset$
$\quad [T(\mathrm{SWIM},\mathrm{ANIMAL}) \equiv T(\mathrm{SWIM},\mathrm{MAMMAL})]]] \supset$
$[[[T(\mathrm{FLY},\mathrm{MAMMAL}) \equiv T(\mathrm{FLY},\mathrm{BAT})] \supset$
$\quad [\tau(\mathrm{FLY},\mathrm{MAMMAL}) \equiv \tau(\mathrm{FLY},\mathrm{BAT})]] \wedge$
$[[T(\mathrm{SWIM},\mathrm{MAMMAL}) \equiv T(\mathrm{SWIM},\mathrm{BAT})] \supset$

$[\tau(\mathrm{SWIM,MAMMAL}) \equiv \tau(\mathrm{SWIM,BAT})]]]$,

which is equivalent to:

$$[\neg T(\mathrm{FLY,MAMMAL}) \wedge [T(\mathrm{SWIM,ANIMAL}) \equiv T(\mathrm{SWIM,MAMMAL})]] \supset \neg T(\mathrm{FLY,MAMMAL}),$$

which is true.

Therefore the above statement is simplified to $T \leq \tau$, assuming $A(T)$. Therefore

$$A(T) \wedge \neg\exists\tau[A(\tau) \wedge \tau < T] \vdash \tau \leq T \wedge T \leq \tau,$$

which means

$$\forall w \forall p[[T(p, last(w)) \equiv T(p,w)] \equiv [\tau(p, last(w)) \equiv \tau(p,w)]],$$

which is equivalent to

$$
\begin{aligned}
&[[T(\mathrm{FLY},\mathrm{ANIMAL}) \equiv T(\mathrm{FLY},\mathrm{BIRD})] \equiv \\
&\quad [\tau(\mathrm{FLY},\mathrm{ANIMAL}) \equiv \tau(\mathrm{FLY},\mathrm{BIRD})]] \wedge \\
&[[T(\mathrm{FLY},\mathrm{BIRD}) \equiv T(\mathrm{FLY},\mathrm{PENGUIN})] \equiv \\
&\quad [\tau(\mathrm{FLY},\mathrm{BIRD}) \equiv \tau(\mathrm{FLY},\mathrm{PENGUIN})]] \wedge \\
&[[T(\mathrm{FLY},\mathrm{ANIMAL}) \equiv T(\mathrm{FLY},\mathrm{MAMMAL})] \equiv \\
&\quad [\tau(\mathrm{FLY},\mathrm{ANIMAL}) \equiv \tau(\mathrm{FLY},\mathrm{MAMMAL})]] \wedge \\
&[[T(\mathrm{FLY},\mathrm{MAMMAL}) \equiv T(\mathrm{FLY},\mathrm{BAT})] \equiv \\
&\quad [\tau(\mathrm{FLY},\mathrm{MAMMAL}) \equiv \tau(\mathrm{FLY},\mathrm{BAT})]] \wedge \\
&[[T(\mathrm{SWIM},\mathrm{ANIMAL}) \equiv T(\mathrm{SWIM},\mathrm{BIRD})] \equiv \\
&\quad [\tau(\mathrm{SWIM},\mathrm{ANIMAL}) \equiv \tau(\mathrm{SWIM},\mathrm{BIRD})]] \wedge \\
&[[T(\mathrm{SWIM},\mathrm{BIRD}) \equiv T(\mathrm{SWIM},\mathrm{PENGUIN})] \equiv \\
&\quad [\tau(\mathrm{SWIM},\mathrm{BIRD}) \equiv \tau(\mathrm{SWIM},\mathrm{PENGUIN})]] \wedge \\
&[[T(\mathrm{SWIM},\mathrm{ANIMAL}) \equiv T(\mathrm{SWIM},\mathrm{MAMMAL})] \equiv \\
&\quad [\tau(\mathrm{SWIM},\mathrm{ANIMAL}) \equiv \tau(\mathrm{SWIM},\mathrm{MAMMAL})]] \wedge \\
&[[T(\mathrm{SWIM},\mathrm{MAMMAL}) \equiv T(\mathrm{SWIM},\mathrm{BAT})] \equiv \\
&\quad [\tau(\mathrm{SWIM},\mathrm{MAMMAL}) \equiv \tau(\mathrm{SWIM},\mathrm{BAT})]].
\end{aligned}
$$

Then, the first conjunct and the second conjunct is true because both sides are false.

The third conjunct is reduced to $\neg T$(FLY,FISH) because $\neg T$(FLY,ANIMAL) is true and right hand side is true.

The fourth conjunct is reduced to $\neg T$(FLY,MAMMAL) because $\neg T$(FLY,ANIMAL) is true and right hand side is true.

The fifth conjunct is true because both sides are false.

The eighth conjunct is reduced to T(SWIM,ANIMAL) because T(SWIM,FISH) is true and right hand side is true.

The sixth conjunct is reduced to T(SWIM,BIRD) because T(SWIM,ANIMAL) is true from the eighth conjunct and right hand side is true.

The seventh conjunct is reduced to T(SWIM,PENGUIN) because T(SWIM,BIRD) is true and right hand side is true.

The ninth conjunct is reduced to T(SWIM,MAMMAL) because T(SWIM,ANIMAL) is true and right hand side is true.

The tenth conjunct is reduced to T(SWIM,BAT) because T(SWIM,MAMMAL) is true and right hand side is true.

Therefore,

$$A(T) \wedge \neg\exists\tau[A(\tau) \wedge \tau < T] \vdash$$
$$\neg T(\text{FLY,FISH}) \wedge \neg T(\text{FLY,MAMMAL}) \wedge$$
$$T(\text{SWIM,ANIMAL}) \wedge T(\text{SWIM,BIRD}) \wedge T(\text{SWIM,MAMMAL}) \wedge$$
$$T(\text{SWIM,PENGUIN}) \wedge T(\text{SWIM,BAT}).$$